

# How continuous quantum measurements in finite dimension are actually discrete

Giulio Chiribella\* and Giacomo Mauro D'Ariano†

QUIT Group, Dipartimento di Fisica “A. Volta” and INFN, via Bassi 6, 27100 Pavia, Italy‡

Dirk Schlingemann§

Institut für Mathematische Physik, Technische Universität Braunschweig,  
Mendelssohnstraße 3, 38106 Braunschweig, Germany

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We show that in finite dimension a quantum measurement with continuous set of outcomes is always equivalent to a continuous random choice of measurements with only finite outcomes.

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When we measure the spin component along a magnetic field with a Stern-Gerlach apparatus, for spin 1/2 particles we have only two possible outcomes: spin up and spin down. This measurement is perfectly repeatable, and can perfectly discriminate between the two orthogonal states  $|\uparrow\rangle$  and  $|\downarrow\rangle$ . It is possible, however, to design an experiment with more than two outcomes, which discriminates optimally—though not perfectly—among three or more non orthogonal states. Indeed, a four-outcome measurement on a two-level system is needed in the eavesdropping of a BB84 cryptographic communication[1], or in the lab to perform an *informationally complete measurement*[2], which determines the quantum state from the measurement statistics.

What about performing a measurement with a *continuous* set of outcomes? This is the case of a measurement designed to optimally determine the “direction” of a spin[3], similarly to what we do in classical mechanics. Such a measurement produces a probability  $p(\mathbf{n})d\mathbf{n}$  of the spin direction falling within the solid angle  $d\mathbf{n}$  around the direction  $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ . Indeed the measurement of direction must be feasible[4]—though in-principle inaccurate—otherwise Quantum Mechanics would fail in describing what we normally observe in the macroscopic world. Actually, this is not the only interesting example of continuous-outcome measurement on a finite-level system: in fact, measurements of this kind have an endless number of applications, e.g. optimal state estimation[5], optimal alignment of directions[6] and reference frames[7], optimal phase estimation[8], and optimal design of atomic clocks[9].

In this Letter we establish a fundamental property of quantum measurements with continuous set of outcomes, namely that for finite level systems any such measurement is equivalent to a continuous random choice of measurements with only *finite* outcomes. This means that any physical quantity measured on a finite dimensional system is intrinsically discrete, while the continuum is pure classical randomness. For a spin 1/2 particle, this fact is well illustrated by the simple observation that the optimal measurement of direction can be equivalently re-

alized by a customary Stern-Gerlach experiment where the magnetic field is randomly oriented. We emphasize that, in general, the discretization of physical quantities does not involve just von Neumann observables, but, more generally, finite measurements with a number of outcomes larger than the Hilbert space dimension. Moreover, using the main result we show that any continuous measurement that optimizes some convex figure of merit (e.g. maximizing the mutual information or the Fisher information, or, alternatively, minimizing a Bayes cost[8, 10]) can be always replaced by a *single* measurement with finite outcomes, without affecting optimality.

Let us start by briefly reviewing the general theoretical description of measurements in Quantum Mechanics. Consider a quantum system (with Hilbert space  $\mathcal{H}$  of dimension  $\dim(\mathcal{H}) = d < \infty$ ), which undergoes a measurement with random outcome  $\omega$ , distributed in the outcome space  $\Omega$ . The probability distribution of the outcomes for any possible state  $\rho$  of the system depends on the specific measuring apparatus used, and is given by a *positive operator-valued measure* (acronym POVM), namely the probability that the outcome falls in the subset  $B \subseteq \Omega$  is given by the Born rule  $p(B) = \text{Tr}[\rho P(B)]$ , where  $P(B)$  is a nonnegative operator with normalization condition  $P(\Omega) = I$ [11]. In the special case in which the measurement is finite, a random result  $i$  from a set of possible outcomes  $\{i = 1, 2, \dots, N\}$  is returned with probability  $p_i = \text{Tr}[\rho P_i]$ ,  $P_i \geq 0$  being nonnegative operators with normalization condition  $\sum_{i=1}^N P_i = I$ .

Before presenting the main result, in order to help intuition, we briefly analyse two simple prototypes of continuous measurement: the optimal measurement of the “spin direction” for a spin 1/2 particle, and the optimal measurement of a phase shift.

The measurement of direction for spin 1/2 particles is given by the POVM [3]

$$P(B) = \int_B \frac{d\mathbf{n}}{2\pi} |\mathbf{n}\rangle\langle\mathbf{n}|, \quad (1)$$

where  $|\mathbf{n}\rangle$  is the eigenvector of  $\mathbf{n} \cdot \mathbf{J}$  with eigenvalue  $+1/2$ ,  $\mathbf{J}$  being the spin operator. It is simple to see that this

measurement is equivalent to the randomization

$$P(B) = \int_{\mathbb{S}^2} \frac{d\mathbf{n}}{4\pi} E^{(\mathbf{n})}(B), \quad (2)$$

where  $d\mathbf{n}/(4\pi)$  is the uniform probability distribution over the unit sphere  $\mathbb{S}^2$  and  $E^{(\mathbf{n})}$  is the POVM

$$E^{(\mathbf{n})}(B) = \chi_B(\mathbf{n})|\mathbf{n}\rangle\langle\mathbf{n}| + \chi_B(-\mathbf{n})|-\mathbf{n}\rangle\langle-\mathbf{n}|. \quad (3)$$

[ $\chi_B(\mathbf{n})$  is the characteristic function of the set  $B$ :  $\chi_B(\mathbf{n}) = 1$  for  $\mathbf{n} \in B$ ,  $\chi_B(\mathbf{n}) = 0$  otherwise.] The POVM  $E^{(\mathbf{n})}$  represents a measurement of direction based on a Stern-Gerlach setup with magnetic field oriented along  $\mathbf{n}$ : if the apparatus outputs “up”, one assigns to the spin the direction  $\mathbf{n}$ , if “down”, one assigns  $-\mathbf{n}$ . With this data-processing, the probability of observing the spin within the region  $B$  is nonzero only if  $B$  contains at least one of the directions  $\pm\mathbf{n}$ . The continuous POVM (1) is then interpreted as a Stern-Gerlach measurement performed with a direction  $\mathbf{n}$  of the magnetic field randomly chosen in the unit sphere.

Another example of continuous measurement is that of phase estimation, where one wants to measure the phase shift  $\phi \in [0, 2\pi)$  experienced by a quantum state under the action of the unitary evolution  $U_\phi = \exp(iN\phi)$ , with  $N = \sum_{n=0}^{d-1} |n\rangle\langle n|$ ,  $\{|n\rangle\}$  orthonormal basis for  $\mathcal{H}$ . The optimal POVM is given by [8]

$$P(B) = \int_B \frac{d\phi}{2\pi} |\phi\rangle\langle\phi|, \quad |\phi\rangle = \sum_{n=0}^{d-1} e^{in\phi} |n\rangle. \quad (4)$$

and is equivalent to the randomization:  $P(B) = \int_0^{2\pi} \frac{d\phi}{2\pi} E^{(\phi)}(B)$ , where  $E^{(\phi)}$  is the POVM  $E^{(\phi)}(B) = \frac{1}{d} \sum_{n=0}^{d-1} \chi_B(\phi_n + \phi) |\phi_n + \phi\rangle\langle\phi_n + \phi|$ ,  $\phi_n = \frac{2\pi n}{d}$ .

We will now show that all continuous measurements in finite dimension can be always interpreted in an analogous way, namely as a continuous random choice of measurements with finite number of outcomes. More precisely, we will prove the following

**Theorem 1** *For any POVM  $P(B)$  the following decomposition holds*

$$P(B) = \int_{\mathcal{X}} dx p(x) E^{(x)}(B), \quad (5)$$

where  $x \in \mathcal{X}$  is a suitable random variable,  $p(x)$  a probability density, and, for every value of  $x$ ,  $E^{(x)}$  denotes a POVM with finite support, i.e. of the form

$$E(B) = \sum_{i=1}^{d^2} \chi_B(\omega_i) P_i \quad (6)$$

$\{\omega_i \in \Omega\}$  being a set of points, and  $\{P_i\}$  being a finite POVM with at most  $d^2$  outcomes[12].

A POVM  $E$  of the form of Eq. (6) is in turn nothing but the continuous data-processing of the finite POVM  $P_i$ , with function of the outcomes  $f(i) = \omega_i$ : if the apparatus outputs  $i$ , then one assigns to the measurement the outcome  $\omega_i$ . The decomposition (5) shows that the continuous POVM  $P$  is achieved by randomly choosing a classical parameter  $x \in \mathcal{X}$  and then performing the finitely-supported POVM  $E^{(x)}$ , depending on  $x$  through the finite POVM  $\{P_i^{(x)}\}$  and through the points  $\{\omega_i^{(x)}\}$ . Operationally, this corresponds to the following recipe: *i)* randomly draw a value of  $x$  according to  $p(x)$ ; *ii)* depending on  $x$ , measure the finite POVM  $P^{(x)}$ , thus getting the outcome  $i$ ; *iii)* for outcome  $i$ , assign to the continuous measurement the outcome  $\omega_i^{(x)}$ . As a first consequence, this simple recipe shows that, contrarily to a rather common belief (see, e.g. [13]), continuous quantum measurements in finite dimension are as feasible as the discrete ones.

The decomposition of the measurement of the “spin direction” given by Eq. (2) provides a concrete example of decomposition (5). In particular, the finitely-supported POVM  $E^{(\mathbf{n})}$  in Eq. (3) is illustrated in Fig. 1 for  $\mathbf{n} = \mathbf{k}$ . Notice that, in general, there may be different randomization schemes yielding the same continuous POVM: as an example, Fig. 1 illustrates another finitely-supported POVM that allows one to reproduce the measurement of direction by simply randomizing the orientation of the Cartesian axes.

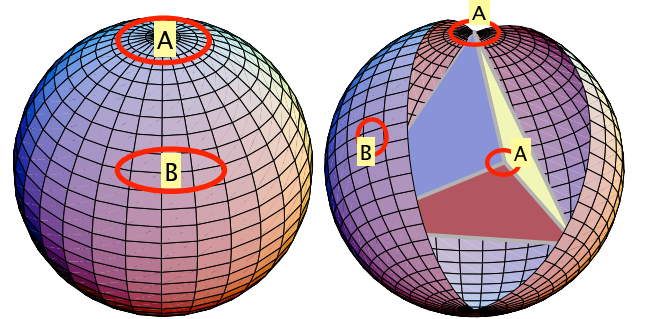


FIG. 1: **Left:** Illustration of the POVM  $E^{(\mathbf{n})}$  in Eq. (3) as an example of finitely-supported POVM  $E^{(x)}$  in Eq. (6). In this specific example the outcome space  $\Omega$  is the unit sphere  $\Omega \equiv \mathbb{S}^2$ , the dimension of the Hilbert space is  $d = 2$ , and only two out of the four terms in Eq. (6) are nonvanishing, namely  $P_1 = |\mathbf{k}\rangle\langle\mathbf{k}|$ ,  $P_2 = |-\mathbf{k}\rangle\langle-\mathbf{k}|$ ,  $P_3 = P_4 = 0$ . The probability of finding the spin direction in a region  $R \subseteq \Omega$  is zero for  $R$  missing the two poles, as  $B$  in the figure, and is possibly nonzero for  $R$  as  $A$ . **Right:** Another example of finitely-supported POVM for  $d = 2$ , corresponding to a SIC (symmetric informationally complete) POVM[14]. The POVM is made of four elements  $P_i$  corresponding to  $\omega_i$  at the vertexes of a tetrahedron. The probability can be nonvanishing only if the region  $R$  contains at least one of these four points, such as in  $A$ , whereas it is always zero in situations as in  $B$ .

We now derive the main result. We fix both the quan-

tum system and the outcome space  $\Omega$ , and consider the set  $\mathcal{P}$  of all possible POVMs for these. This is a convex set, since given any two POVMs  $P'$  and  $P''$ , their *convex combination*  $P^{(\lambda)} = \lambda P' + (1 - \lambda)P''$  for  $\lambda \in [0, 1]$  is still a POVM, namely the whole segment joining  $P'$  and  $P''$  is contained in  $\mathcal{P}$ . The *extremal points* of the convex set  $\mathcal{P}$  are those POVMs that cannot be written as convex combination of two different POVMs. Stated differently, a POVM  $P \in \mathcal{P}$  is not extremal if and only if it is the midpoint of a segment completely contained in  $\mathcal{P}$ , i.e. if and only if there exist two distinct points  $P', P'' \in \mathcal{P}$ ,  $P' \neq P''$  such that  $P = \frac{1}{2}(P' + P'')$ . This is equivalent to the existence of a direction  $Q \neq 0$  and a positive number  $\epsilon > 0$  such that  $P + tQ \in \mathcal{P}$  for any  $t \in [-\epsilon, \epsilon]$ . The standard name for the direction  $Q$  in convex analysis is *perturbation*. Here the perturbation  $Q$  is a function that associates to any subset  $B \subseteq \Omega$  an operator  $Q(B)$ , fulfilling the three requirements: *i*)  $Q(B)$  is Hermitian for any subset  $B \subseteq \Omega$ ; *ii*)  $Q(\Omega) = 0$ ; *iii*)  $P(B) + tQ(B) \geq 0$  for any  $B \in \Omega$  and for any  $t \in [-\epsilon, \epsilon]$ .

If there exists a nonzero perturbation  $Q$  for  $P$ , then  $P$  is non extremal: using this criterion, we now establish that the extremal POVMs must necessarily have finite support, namely they must be of the form of Eq. (6).

The proof takes advantage of the following:

**Lemma 1** *Every POVM  $P \in \mathcal{P}$  admits a density with unit trace, namely for any POVM  $P$  there exists a finite measure  $\mu(d\omega)$  over  $\Omega$  such that*

$$P(B) = \int_B \mu(d\omega) M(\omega), \quad (7)$$

with  $M(\omega) \geq 0$  and  $\text{Tr}[M(\omega)] = 1$   $\mu$ -almost everywhere.

**Proof.** Consider the finite measure  $\mu(d\omega)$  defined by  $\mu(B) = \text{Tr}[P(B)]$ ,  $\forall B \subseteq \Omega$ . Since  $P(B) \geq 0$ , one has  $P(B) \leq \text{Tr}[P(B)] I = \mu(B) I$ , namely  $P(B)$  is dominated by the measure  $\mu(B)$ . This implies that  $P$  admits a density  $M(\omega)$  with respect to  $\mu(d\omega)$ . Clearly, the density  $M(\omega)$  has to be nonnegative  $\mu$ -almost everywhere. Moreover, for any  $B \subseteq \Omega$  one has  $\int_B \mu(d\omega) \equiv \mu(B) = \text{Tr}[P(B)] = \int_B \mu(d\omega) \text{Tr}[M(\omega)]$ , whence  $\text{Tr}[M(\omega)] = 1$   $\mu$ -a.e. ■

Thanks to this Lemma, we can represent any POVM  $P \in \mathcal{P}$  using its density  $M(\omega)$  as in Eq. (7). To prove that an extremal POVM must be of the form (6) it is enough to show that for extremal POVMs the measure  $\mu(d\omega)$  must be concentrated on a finite set of outcomes  $\{\omega_1, \dots, \omega_{d^2}\}$ , i.e.  $\mu(B) = 0$  for any set  $B \subseteq \Omega$  not containing anyone of the points  $\omega_i$ . We recall the definition of the support of a measure  $\mu(d\omega)$  as the set of all points  $\omega \in \Omega$  such that  $\mu(B) > 0$  for any open set  $B$  containing  $\omega$ .

**Lemma 2** *Let  $P \in \mathcal{P}$  be a POVM and  $\mu(d\omega)$  the measure defined by  $\mu(B) = \text{Tr}[P(B)]$ . If  $P$  is extremal, then*

*the support of  $\mu(d\omega)$  is finite and contains no more than  $d^2$  points.*

**Proof.** Suppose that the support contains more than  $d^2$  points. In this case, one can take  $d^2 + 1$  points  $\omega_i \in \Omega$  in the support and  $d^2 + 1$  disjoint open sets  $U_i \subset \Omega$ ,  $i = 1, \dots, d^2 + 1$ , such that  $\omega_i \in U_i$  for any  $i$  [15]. As a consequence, the space  $L^\infty(\Omega, \mu)$  of integrable functions  $f(\omega)$  that are bounded  $\mu$ -almost everywhere has dimension at least  $d^2 + 1$  (indeed, the characteristic functions  $\chi_{U_i}(\omega)$  are a set of  $d^2 + 1$  bounded and linearly independent functions). Then, consider the matrix elements  $f_{ij}(\omega) = \langle i|M(\omega)|j \rangle$ , where  $M(\omega)$  is the POVM density of Eq. (7), and  $|i\rangle, |j\rangle$  are elements of an orthonormal basis for  $\mathcal{H}$ . Since the operators  $M(\omega)$  are nonnegative with unit trace a.e., the functions  $f_{ij}(\omega)$  are bounded a.e., namely  $f_{ij} \in L^\infty(\Omega, \mu) \forall i, j$ . Moreover, since the space  $L^\infty(\Omega, \mu)$  has dimension larger than  $d^2$ , it must contain at least one function  $g(\omega) \neq 0$  that is linearly independent from the set  $\{f_{ij}\}$ . Using the Gram-Schmidt orthogonalization procedure, such a function  $g$  can be always chosen to be orthogonal to all  $f_{ij}$ , namely  $\int_\Omega \mu(d\omega) g^*(\omega) f_{ij}(\omega) = 0 \quad \forall i, j$ . Finally, since  $f_{ij}^*(\omega) = f_{ji}(\omega) \quad \forall i, j$ , such a  $g$  can be also chosen to be real. Now, we claim that the Hermitian operators  $Q(B)$  defined by

$$Q(B) = \int_B \mu(d\omega) g(\omega) M(\omega) \quad (8)$$

provide a perturbation for the POVM  $P$ . Indeed, we have  $Q(\Omega) = 0$  as the matrix elements  $\langle i|Q(\Omega)|j \rangle$  are zero for any  $i, j$ :

$$\begin{aligned} \langle i|Q(\Omega)|j \rangle &= \int_\Omega \mu(d\omega) g(\omega) \langle i|M(\omega)|j \rangle \\ &= \int_\Omega \mu(d\omega) g(\omega) f_{ij}(\omega) = 0. \end{aligned} \quad (9)$$

Moreover, since  $g \in L^\infty(\Omega, \mu)$ , there exists a positive number  $c$  such that  $|g(\omega)| \leq c < \infty$  a.e., thus implying that the operators  $M(\omega) [1 + tg(\omega)]$  are a.e. nonnegative for any  $t \in [-\epsilon, \epsilon]$ , with  $\epsilon = 1/(2c)$ . Hence, integrating over any subset  $B$ , we obtain that the operators  $P(B) + tQ(B)$  are nonnegative, namely  $Q$  is a perturbation. Finally,  $Q$  is nonzero, otherwise taking the trace of Eq. (8), and using that  $\text{Tr}[M(\omega)] = 1$  a.e. we would get  $0 = \text{Tr}[Q(B)] = \int_B \mu(d\omega) g(\omega) \quad \forall B$ , thus implying  $g = 0$ , which is not possible by definition of  $g$ . In conclusion, if the support of  $\mu(d\omega)$  contains more than  $d^2$  points, then the POVM  $P$  has a nonzero perturbation, whence it is not extremal. ■

Lemma 2 establishes that an extremal POVM has necessarily the form of Eq. (6), namely it can be realized by measuring a finite POVM  $P_i$  and declaring measurement outcome  $\omega_i$ . Using this fact, we readily obtain the proof of the main Theorem:

**Proof of Theorem 1.** Due to the standard Krein-Milman theorem of convex analysis, any point of a compact convex set is a continuous convex combination of points that are either extremal or limit of extremals. On the other hand, it is simple to prove that the set  $\mathcal{P}$  of all POVMs is compact[16], and that any limit of extremal POVMs is still a POVM of the form (6)[17]. For a detailed mathematical proof see Ref. [18]. ■

We now want to explore some consequences of decomposition (5) for optimization of POVM's and for Quantum Tomography. Optimizing a quantum measurement consists in finding the POVM  $P$  that maximizes the value of a figure of merit  $\mathcal{F}[P]$ —e. g. mutual or Fisher information, average fidelity, or any Bayes gain. In all these cases  $\mathcal{F}[P]$  is convex, i. e.  $\mathcal{F}[\lambda P' + (1 - \lambda)P''] \leq \lambda \mathcal{F}[P'] + (1 - \lambda)\mathcal{F}[P'']$  for any  $\lambda \in [0, 1]$ . Suppose now that a continuous POVM  $P$  is optimal for  $\mathcal{F}$ . Combining convexity of  $\mathcal{F}$  with Eq. (5) one has

$$\mathcal{F}_{\max} = \mathcal{F}[P] \leq \int_{\mathcal{X}} dx p(x) \mathcal{F}[E^{(x)}] \leq \mathcal{F}_{\max}, \quad (11)$$

which implies  $\mathcal{F}[E^{(x)}] = \mathcal{F}_{\max}$  for any  $x$  except at most a set of zero measure. This means that all the finite POVMs  $E^{(x)}$  are equally optimal: in particular, *for any optimal continuous measurement there is always an optimal measurement with finite (no more than  $d^2$ ) number of outcomes.* In special situations some explicit algorithms to find optimal finite measurements are known[13, 19, 20]. In particular, Ref.[19] shows that in many cases the minimal number of outcomes is larger than  $d = \dim(\mathcal{H})$ . Combined with the above result, this fact definitely proves that the quantum discretization cannot rely solely on von Neumann measurements.

Regarding Quantum Tomography, using the present analysis we can make mathematically precise the common intuition that an informationally-complete continuous measurement is equivalent to a Tomography scan made of a random choice of observables—or more generally POVMs. Indeed one can estimate the ensemble average of any operator  $A$  by using the two data processing  $f_A(\omega)$  and  $f_A(\omega_i^{(x)})$  for continuous POVM and tomography, respectively, as follows

$$A = \int_{\Omega} d\omega f_A(\omega) M(\omega) = \int_{\mathcal{X}} dx p(x) \sum_{i=1}^{d^2} f_A(\omega_i^{(x)}) P_i^{(x)}.$$

In conclusion, in this Letter we showed that continuous quantum measurements can be always realized by performing finite measurements depending on a random classical parameter. Physical properties, such as spatial orientation and time, are then intrinsically discrete when measured on finite level quantum systems.

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\* Electronic address: chiribella@fisicavolta.unipv.it

† Electronic address: dariano@unipv.it

‡ URL: <http://www.qubit.it>

§ Electronic address: d.schlingemann@tu-bs.de

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- [17] For the easy case of  $\Omega$  compact (e.g. spin direction or phase), let  $E^{(n)}(B) = \sum_{i=1}^{d^2} \chi_B(\omega_i^{(n)}) P_i^{(n)}$  be a sequence of extremal POVMs with limit  $E(B)$ . For any  $i$  we can find a subsequence  $\omega_i^{(n_k)}$  converging to a point  $\omega_i \in \Omega$ . Similarly, we can choose  $n_k$  so that also the finite POVMs  $\{P_i^{(n_k)}\}$  converge to a finite POVM  $\{P_i\}$ . Hence  $E(B) = \sum_{i=1}^{d^2} \chi_B(\omega_i) P_i$ , namely the limit is still of the form (6).
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